Bound on genuine multipartite correlations from the principle of information causality

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Quantum mechanics is not the unique no-signaling theory which is endowed with stronger-than-classical correlations, and there exists a broad class of no-signaling theories allowing even stronger-than-quantum correlations. The principle of information causality has been suggested to distinguish quantum theory from these nonphysical theories, together with an elegant information-theoretic proof of the quantum bound of two-particle correlations. In this work, we extend this to genuine N-particle correlations that cannot be reduced to mixtures of states in which a smaller number of particles are entangled. We first express Svetlichny's inequality in terms of multipartite no-signaling boxes, then prove that the strongest genuine multipartite correlations lead to the maximal violation of information causality. The maximal genuine multipartite correlations under the constraint of information causality is found to be equal to the quantum mechanical bound. This result consolidates information causality as a physical principle defining the possible correlations allowed by nature, and provides intriguing insights into the limits of genuine multipartite correlations in quantum theory.

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Introduction The violation of Bell inequalities [1, 2] proves that the quantum mechanics cannot be regarded as a local realistic theory. Tsirelson [3] proved an upper bound on the violation of the CHSH inequality [2], which means that the amount of non-locality allowed by quantum mechanics is limited. One may think that Tsirelson bound is a consequence of relativity, but Popescu and Rohrlich [4] showed that there exists a broad class of no-signaling theories which allow even stronger-than-quantum correlations. An example of the no-signaling theories is the Popescu and Rohrlich boxes (PR-boxes) [4]. This broad class of no-signaling theories possessing extremely powerful correlations are usually called post-quantum theories and modeled as no-signaling boxes (NS-boxes) [5]. These post-quantum theories have much in common with quantum mechanics, such as no-cloning [6], no-broadcasting [7], monogamy of correlations [6], information-disturbance trade-offs [8], and the security for key distribution [9], so there is a need to find some principles at the very root of quantum theory and distinguish it from these post-quantum theories. In recent years, an intensive study has been made on this issue. In Ref. [10], van Dam showed that the availability of PR-boxes makes communication complexity trivial. However, communication complexity is not trivial in quantum physics and it is strongly believed that devices producing such correlations making communication complexity trivial are very unlikely to exist. Later, Brassard *et al.* proved that some post-quantum theories would lead to an implausible simplification of distributed computational tasks [11–13]. More recently, Barnum *et al.* [14, 15] showed that the combination of local quantum measurement assumption and relativity results in quantum correlations, and in Ref. [15] the authors provided a unified framework for all no-signaling theories.

From an information theoretic point of view, Pawłowski *et al.* [16] suggested a bold physical principle: information causality (IC), stating that communication of n classical bits causes information gain of at most n bits. When $n = 0$ IC is just the no-signaling principle. In a bipartite scenario where each party has two inputs and two outputs, Pawłowski *et al.* showed that IC is respected both in classical and quantum physics, but all correlations stronger than the strongest quantum correlations (Tsirelson bound) violate it, and they derived Tsirelson bound from IC. It must be noted that there are some stronger-than-quantum correlations which are not known to violate IC [17, 18].

The present work is to extend the research of understanding the quantum mechanical bound on nonlocal correlations to genuine multipartite correlations. The structure of multipartite correlations is much richer than that of bipartite correlations [5]. For example, in [19] the authors dealt with a tripartite scenario where each party has two inputs and two outputs, they found that there exist 53856 extremal no-signaling tripartite correlations which belong to 46 inequivalent classes, and there are more than three classes which feature genuine tripartite nonlocality. So there exist many inequivalent types of genuine multipartite correlations, and in the present paper we deal only with Svetlichny genuine multipartite correlations which is relevant to Svetlichny's inequality (SI) [20, 21]. We first express SI in terms of multipartite no-signaling boxes, and then prove that the strongest Svetlichny genuine multipartite correlation leads to the maximal violation of IC. Under the constraint of IC, the maximal Svetlichny genuine multipartite correlation just equals to the quantum mechanical bound.

Tripartite Svetlichny's inequality We first introduce SI of three-particle [20], which can distinguish between gen-

uine three-particle correlations and two-particle correlations. A violation of SI implies the presence of genuine three-particle correlations. Consider three observers, Alice, Bob, and Carol, who share three entangled qubits. Each of the three observers can choose to measure one of two dichotomous observables. We denote $x \in \{0,1\}$ and $A \in \{-1,1\}$ as Alice's measurement choice and outcome respectively, and similarly y and B (z and C) for Bob's (Carol's). Thus SI can be expressed as [20]

$$\begin{aligned} S \;\equiv\; & |E(ABC|x=0,y=0,z=0) \\ & +E(ABC|x=0,y=0,z=1) \\ & +E(ABC|x=0,y=1,z=0) \\ & +E(ABC|x=1,y=0,z=0) \\ & -E(ABC|x=0,y=1,z=1) \\ & -E(ABC|x=1,y=0,z=1) \\ & -E(ABC|x=1,y=1,z=0) \\ & -E(ABC|x=1,y=1,z=1)| \leq 4, \end{aligned} \tag{1}$$

where $E(ABC|x,y,z)$'s represent the expectation value of the product of the measurement outcomes of the observables x, y, and z, and we call S as Svetlichny operator. It was shown by Svetlichny [20] that quantum predictions violate his inequality, and the maximum violation ($S = 4\sqrt{2}$) allowed in quantum mechanics can be achieved with GHZ states [22].

If we define $a = \frac{1-A}{2}$, $b = \frac{1-B}{2}$, and $c = \frac{1-C}{2}$, each of $E(ABC|x,y,z)$'s can be expressed in terms of probabilities, for example,

$$\begin{aligned} & E(ABC|x=0,y=0,z=0) \\ = \; & 2P(a\oplus b\oplus c = xy\oplus yz\oplus xz|x=0,y=0,z=0)-1, \\ & E(ABC|x=0,y=1,z=1) \\ = \; & 1-2P(a\oplus b\oplus c = xy\oplus yz\oplus xz|x=0,y=1,z=1), \end{aligned} \tag{2}$$

where $P(a\oplus b\oplus c = xy\oplus yz\oplus xz|x=0,y=0,z=0)$ is the probability that $a\oplus b\oplus c = xy\oplus yz\oplus xz$ under the condition $x=0, y=0, z=0$, and $\oplus$ denotes the addition modulo 2. So we can also write the SI as

$$\frac{1}{8}\sum_{x,y,z} P(a\oplus b\oplus c = xy\oplus yz\oplus xz|x,y,z) \leq \frac{3}{4}. \tag{3}$$

From the above inequality we find that there is a convenient way of thinking about genuine three-particle correlations by three black boxes shared by Alice, Bob, and Carol. The correlations between inputs x, y, z and outcomes a, b, c are described by probability $P(a\oplus b\oplus c = xy\oplus yz\oplus xz|x,y,z)$, and we call these boxes Svetlichny boxes [5]. The maximal algebraic value $S = 8$ is reached if and only if $P(a\oplus b\oplus c = xy\oplus yz\oplus xz|x,y,z) = 1$ for any x, y, and z. It is obvious that Svetlichny boxes belong to tripartite NS-boxes, since Svetlichny boxes still satisfy the principle of no-signaling due to uniformly random local outcomes.

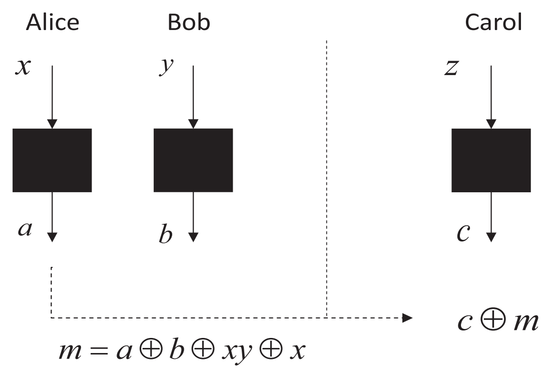

FIG. 1: Alice, Bob, and Carol shared Svetlichny boxes ($P(a\oplus b\oplus c = xy\oplus yz\oplus xz|x,y,z)$), Alice and Bob sit next to each other, at a long distance from Carol. Alice or Bob sends message $m = a\oplus b\oplus xy\oplus x$ to Carol, if input $z=0$ Carol wants to learn x, and if input $z=1$ she wants to learn y. Upon receiving message m from Alice and Bob, Carol can compute her guess $g = c\oplus m = a\oplus b\oplus c\oplus xy\oplus x$. The probabilities of correctly guessing x and y are Eq. (6) and Eq. (7). With the help of suitable Svetlichny boxes these three persons can violate IC.

Svetlichny boxes lead to violation of IC Before elucidating that Svetlichny boxes can maximally violate IC, we first give a brief overview of IC. Suppose there are two persons, Alice and Bob, Alice has N random and independent bits $(a_1, a_2, ..., a_N)$, and Bob receives a random variable $l \in \{1,2,...,N\}$. Alice can send n classic bits to Bob, and Bob's task is to guess the value of the l-th bit in Alice's list with the help of the n bits. The amount of the information about Alice's list gained by Bob is measured by

$$I \equiv \sum_{k=1}^{N} I(a_k : g|l=k) \geq N - \sum_{k=1}^{N} h(p_k), \tag{4}$$

where $I(a_k : g|l=k)$ is Shannon mutual information between a_k and g (g is Bob's guess), and p_k is the probability that $a_k = g$, both computed in the case of that Bob has received $l=k$. In Eq. (4), the inequality can be proved by Fano inequality [23]. IC states that physically allowed theories must have

$$I \leq n. \tag{5}$$

Now we consider that there exist Svetlichny boxes shared by Alice, Bob, and Carol (see Fig.(1)). Alice and Bob sit next to each other, at a long distance from Carol, and Alice(Bob) can send one bit to Carol. Carol's mission is to guess the value of x (Alice's input) when she receives $z=0$ and guess the value of y (Bob's input) when she receives $z=1$. The message which sent by Alice (Bob) to Carol is $m = a\oplus b\oplus xy\oplus x$. Upon

receiving the message m, Carol can compute her guess $g = c \oplus m = a \oplus b \oplus c \oplus xy \oplus x$. The probabilities of correct guess of x and y are

$$p_x = \frac{1}{4}[P(a \oplus b \oplus c = 0|0,0,0) + P(a \oplus b \oplus c = 0|0,1,0) + P(a \oplus b \oplus c = 0|1,0,0) + P(a \oplus b \oplus c = 1|1,1,0)] \quad (6)$$

$$p_y = \frac{1}{4}[P(a \oplus b \oplus c = 0|0,0,1) + P(a \oplus b \oplus c = 1|0,1,1) + P(a \oplus b \oplus c = 1|1,0,1) + P(a \oplus b \oplus c = 1|1,1,1)] \quad (7)$$

The Svetlichny boxes of $P(a \oplus b \oplus c = xy \oplus yz \oplus xz|x,y,z) = 1$ predict $p_x = p_y = 1$, from Eq. (4) we have $I = 2$ for $n = 1$, so the Svetlichny boxes can maximally violate IC.

The bound on genuine three-particle correlations Now we proceed to show that stronger-than-quantum genuine three-particle correlations lead to the violation of IC.

Since it is known that the maximal violation is obtained by the GHZ state [21] and in this case all probabilities $P(a \oplus b \oplus c = xy \oplus yz \oplus xz|x,y,z)$ are the same, it is a natural choice to consider the isotropic boxes and indeed this choice successfully leads to the quantum bound. The isotropic Svetlichny boxes can be written as

$$P(a \oplus b \oplus c = xy \oplus yz \oplus xz|x,y,z) = \frac{1+E}{2}, \quad (8)$$

where $0 \leq E \leq 1$. The Svetlichny boxes of Eq. (8) has strongest genuine tripartite correlations when $E = 1$, and it correspond to uncorrelated random bits when $E = 0$. SI of Eq. (3) is violated as soon as $E > \frac{1}{2}$, and the quantum bound $S = 4\sqrt{2}$ corresponds to $E = \frac{\sqrt{2}}{2}$.

In Fig(2), we illustrate how to transform Svetlichny boxes to bipartite NS-boxes. If the initial Svetlichny boxes are described by probability $P(a \oplus b \oplus c = xy \oplus yz \oplus xz|x,y,z) = \frac{1+E}{2}$, the transformed bipartite NS-boxes can be described by probability $P(A \oplus c = (x \oplus y)z|x,y,z) = \frac{1+E}{2}$. So any bipartite NS-boxes of $P(a \oplus b = xy|x,y) = \frac{1+E}{2}$ can be simulated by Svetlichny boxes of $P(a \oplus b \oplus c = xy \oplus yz \oplus xz|x,y,z) = \frac{1+E}{2}$. In Ref. [16], the authors proved that the bipartite NS-boxes of $P(a \oplus b = xy|x,y) = \frac{1+E}{2}$ would lead to the violation of IC as soon as $E > \frac{\sqrt{2}}{2}$, thus we can conclude that the Svetlichny boxes of $P(a \oplus b \oplus c = xy \oplus yz \oplus xz|x,y,z) = \frac{1+E}{2}$ lead to the violation of IC as soon as $E > \frac{\sqrt{2}}{2}$. So we have proven that the maximal genuine three-particle correlation under the constraint of IC just corresponds to the quantum bound of violation of SI.

The bound on genuine multipartite correlations In Ref. [21], the SI of three-particle has been generalized to the case of N particles. Here, by using the derivation method of Eq. (3) we express these N-particle SI in terms of

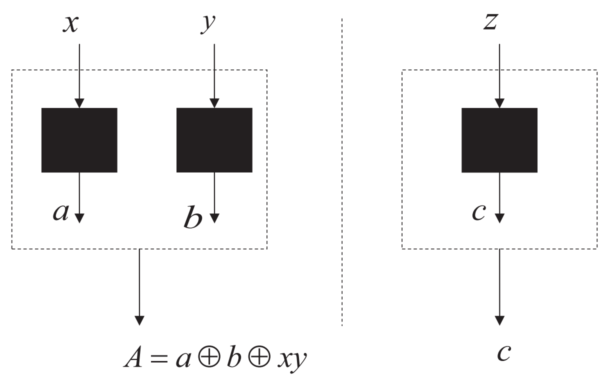

FIG. 2: We can transform Svetlichny boxes to bipartite NS-boxes. The left two boxes are combined to form a new box, the right box is unchanged. The only difference is that there are two input bits on the one side of this transformed NS-boxes, while there is only one input bit on both sides of "normal" bipartite NS-boxes. If the initial Svetlichny boxes is described by probability $P(a \oplus b \oplus c = xy \oplus yz \oplus xz|x,y,z) = \frac{1+E}{2}$, the transformed bipartite NS-boxes can be described by probability $P(A \oplus c = (x \oplus y)z|x,y,z) = \frac{1+E}{2}$.

probability, then genuine N-particle correlations can be modeled as N-particle no-signaling boxes (NNS-boxes). Suppose there are N players who shared N particles, each one of them performs dichotomous measurements on each of the N particles. The measurement settings are represented by x_1, x_2,...x_N respectively, with possible values $0, 1$. The measurement results are represented by a_1, a_2,...a_N respectively, and also with possible values $0, 1$. Then the N-particle SI can be written as (proof in the Appendix)

$$\frac{1}{2^N} \sum_{\{x_i\}} P\Big(\sum_i^N a_i = \sum_{i<j\leq N} x_i x_j | x_1, x_2, ..., x_N\Big) \leq \frac{3}{4}, \quad (9)$$

where $\{x_i\}$ stands for an N-tuple $x_1, ..., x_N$, $\sum_i^N$ and $\sum_{i<j\leq N}$ both denote summation modula 2, and P is the probability that $\sum_i^N a_i = \sum_{i<j\leq N} x_i x_j$ with given $x_1, x_2, ..., x_N$. The isotropic NNS-boxes can be written as a simple form:

$$P\Big(\sum_i^N a_i = \sum_{i<j\leq N} x_i x_j | x_1, x_2, ..., x_N\Big) = \frac{1+E}{2}, \quad (10)$$

where $0 \leq E \leq 1$. SI of Eq. (10) is violated as soon as $E > \frac{1}{2}$. The quantum bound of genuine N-particle correlations corresponds to $E = \frac{\sqrt{2}}{2}$, and it can be achieved with N-particle GHZ states [21].

In Fig.(3), we illustrate the transformation of NNS-boxes to bipartite NS-boxes. If the initial NNS-boxes is described by probability $P\Big(\sum_i^N a_i = \sum_{i<j\leq N} x_i x_j | x_1, x_2, ..., x_N\Big) = \frac{1+E}{2}$, the transformed

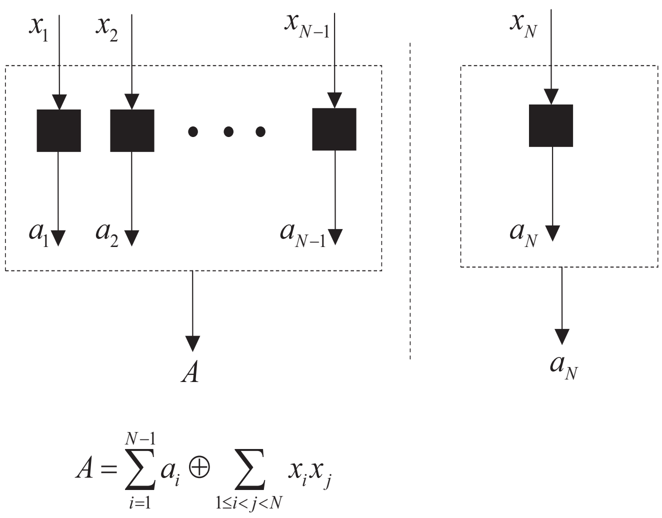

FIG. 3: We can transform NNS-boxes to bipartite NS-boxes. The left $N-1$ boxes are combined to form a new box, the right box is unchanged. The only difference is that there are $N-1$ input bits on the one side of this transformed NS-boxes, while there is only one input bit on both sides of "normal" bipartite NS-boxes. If the initial NNS-boxes is described by probability $P\Big(\sum_i^N a_i = \sum_{i<j\le N} x_i x_j | x_1, x_2, ..., x_N\Big) = \frac{1+E}{2}$, the transformed bipartite NS-boxes can be described by probability $P(A \oplus a_N = (x_1 \oplus x_2 \oplus ... \oplus x_{N-1})x_N) = \frac{1+E}{2}$.

bipartite NS-boxes can be described by probability $P(A \oplus a_N = (x_1 \oplus x_2 \oplus ... \oplus x_{N-1})x_N) = \frac{1+E}{2}$. So any bipartite NS-boxes of $P(a \oplus b = xy|x, y) = \frac{1+E}{2}$ can be simulated by NNS-boxes of $P\Big(\sum_i^N a_i = \sum_{i<j\le N} x_i x_j | x_1, x_2, ..., x_N\Big) = \frac{1+E}{2}$. This implies that the NNS-boxes of $P\Big(\sum_i^N a_i = \sum_{i<j\le N} x_i x_j | x_1, x_2, ..., x_N\Big) = \frac{1+E}{2}$ would lead to the violation of IC as soon as $E > \frac{\sqrt{2}}{2}$. So we have proven that the maximal genuine N-particle correlations under the constraint of IC just corresponds to the quantum bound of violation of SI of N-particle.

Discussion In this work we give an information-theoretical proof about the quantum bound of violations of SI, i.e. the maximal violations of SI just equal to the quantum bound due to the constraint of IC. We first employ a genuine multipartite correlation resource to simulate a bipartite correlation, and then make use of the previously known bipartite results [16]. We note that, while there exist many different protocols to simulate a bipartite correlation by using a genuine N-partite correlation, all the simulations will result in the same conclusion: if there exists a stronger-than-quantum genuine N-partite correlation then we can use it to simulate a bipartite correlation which can breach IC. With regard to different simulation protocols, for example, we can combine left k boxes to form a new box and the remaining $N-k$ boxes are combined to form the other new box. If the initial N-partite no-signaling boxes is described by probability $P\Big(\sum_i^N a_i = \sum_{i<j\le N} x_i x_j | x_1, x_2, ..., x_N\Big) = \frac{1+E}{2}$, then the transformed bipartite no-signaling boxes is described by probability $P(A \oplus B = (x_1 \oplus x_2 \oplus ... \oplus x_k)(x_{k+1} \oplus ... \oplus x_N)) = \frac{1+E}{2}$, where $A = \sum_{i=1}^k a_i \oplus \sum_{1\le i<j\le k} x_i x_j$ and $B = \sum_{i=k+1}^N a_i \oplus \sum_{k+1\le i<j\le N} x_i x_j$. This simulation is different from the previous simulation but would lead to the same conclusion.

The genuine multipartite correlations are essentially more powerful correlation resources than bipartite correlations. One Svetlichny box can simulate a PR-box, but we must use three PR-boxes to simulate a Svetlichny box [5]. So the bound of genuine multipartite correlations what the IC tells us is a genuine new and exciting result, which bears some fundamental differences from the known bipartite results.

Appendix Proof of inequality (9)

Suppose there are N players who shared N particles, each one of them performs dichotomous measurements on each of the N particles. The measurement settings are represented by x_1, x_2,...x_N, respectively, with possible values $0, 1$. The measurement results are represented by A_1, A_2,...A_N, respectively, and with possible values $-1, 1$. Then the original N-particle SI [21] can be expressed as

$$\begin{aligned} S_N \equiv & \,|\sum_{\{x_i\}} v(x_1, x_2, ..., x_N) E(A_1 A_2 \cdots A_N | x_1, x_2, ..., x_N)| \\ \le & \; 2^{N-1}, \end{aligned} \quad (11)$$

where $\{x_i\}$ stands for an N-tuple $x_1, ..., x_N$, $E(A_1 A_2 \cdot \cdot A_N | x_1, x_2, ..., x_N)$ represents the expectation value of the product of the measurement outcomes of the observables $x_1, x_2, ..., x_N$, and $v(x_1, x_2, ..., x_N)$ is a sign function given by

$$v(x_1, x_2, ..., x_N) = (-1)^{[\frac{k(k-1)}{2}]}, \quad (12)$$

where k is the number of times index 1 appears in $(x_1, x_2, ..., x_N)$.

We can easily find that

$$v(x_1, x_2, ..., x_N) = (-1)^{\sum_{i<j\le N} x_i x_j}, \quad (13)$$

where $\sum_{i<j\le N}$ denotes summation modula 2.

If we define $a_i = \frac{1-A_i}{2}$, then

$$\begin{aligned} & E(A_1 A_2 \cdots A_N | x_1, x_2, ..., x_N) = (-1)^{\sum_{i<j\le N} x_i x_j} \\ & \cdot \Big[2P\Big(\sum_i^N a_i = \sum_{i<j\le N} x_i x_j | x_1, x_2, ..., x_N\Big) - 1\Big], \end{aligned} \quad (14)$$

where $\sum_i^N$ denotes summation modula 2. From Eq. (11) and Eq. (14) we finally obtain inequality (9) in the main text.

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